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LETTER TO THE EDITOR

The excitations of the symplectic integrable models and their applications

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Abstract
The Bethe ansatz equations of the fundamental $Sp(2N)$ integrable model are solved by a peculiar configuration of roots leading us to determine the nature of the excitations. They consist of N elementary generalized spinons and $N-1$ composite excitations made by special convolutions between the spinons. This fact is essential to determining the low-energy behaviour which is argued to be described in terms of $2N$ Majorana fermions. Our results have practical applications to spin–orbital systems and also shed new light on the connection between integrable models and Wess–Zumino–Witten field theories.

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The study of quantum one-dimensional integrable models has turned out to be a fruitful venture since the seminal work of Bethe in 1931 [1]. Over the years, solvable models have been extremely useful in many subfields of physics, providing us with a rich laboratory in which new theoretical insights and non-perturbative methods can readily be tested. Recent progress in the experimental study of low-dimensional materials, e.g. spin ladders and carbon nanotubes [2], has been an additional source of motivation to investigate one-dimensional exactly solvable models.

The basic concept of quantum integrability is the S-matrix which represents either the factorized scattering of particles of $(1+1)$ quantum field theories or the statistical weights of integrable two-dimensional lattice models. It turns out that the symmetry of the S-matrix plays a fundamental role in the theory and classification of integrable systems [3, 4]. Of particular interest is the $Sp(2N)$ symmetry which preserves bilinear antisymmetric metrics, typical of systems with N-component Dirac fermions. Even though the Bethe ansatz solution of the integrable $Sp(2N)$ models has long been known [5, 6], basic properties such as the nature of the elementary excitations and the low-energy behaviour have not yet been determined.

The purpose of this letter is to unveil the physical content of the fundamental (vector representation) $Sp(2N)$ solvable magnet. We argue that the low-energy properties are given in terms of $2N$ Majorana fermions due to the presence of special low-lying excitations in the spectrum. There exist at least two immediate applications of this result. First, it provides us with a unique counter-example to the conjecture that integrable models based on the vector representation of Lie algebras should be lattice realizations of Wess–Zumino–Witten (WZW) conformal theories [7–9]. In fact, we predict $c = N$ for the fundamental integrable $Sp(2N)$ model while the central charge of the $Sp(2N)$ WZW theory is $c = N(2N+1)/(N+2)$. Next, our study is of utility for one-dimensional systems with coupled spin and orbital degrees of freedom such as the spin–orbital [10–12] and spin–tube [13] models. More precisely, we recall that the effective spin–isospin Hamiltonian describing these systems may be written in the form [10, 14]

$$H_{SO}(J_0, J_1, J_2) = \sum_{i=1}^{L} \sum_{\alpha=0}^{2} J_\alpha P_{i,i+1}^{(\alpha)} \tag{1}$$

where J_α are superexchange constants and $P_{i,i+1}^{(\alpha)}$ denote the respective projections on the singlet, triplet and doublet spin–isospin states, see [14] for details. Writing these projectors in terms of two commuting sets of Pauli matrices, it is not difficult to identify that the integrable $Sp(4)$ spin chain [17] corresponds to the point $J_0/J_1 = J_0/J_2 = 1/3$. This point is interesting because it corresponds to both anisotropic and asymmetric spin-1/2 couplings[1], thus being closer to representing the properties of realistic materials [18, 19] than the integrable $SU(4)$ case $J_0 = J_1 = J_2$ [16]. This then provides us with a rare opportunity to determine exactly the nature of the excitations in a relevant spin–orbital model.

In the context of statistical mechanics the integrable $Sp(2N)$ model is a multistate vertex system defined on the square lattice whose bond variables take $2N$ possible values. With each type of configuration of four bonds a, b, c, d meeting at a vertex, we associate a Boltzmann weight factor $S_{ab}^{cd}(\lambda)$ where λ is the spectral parameter. Compatibility between integrability and the $Sp(2N)$ invariance ('ice-type' restriction) leads us to the following amplitudes [4, 6]:

$$S_{ab}^{cd}(\lambda) = \delta_{a,d}\delta_{c,b} + \lambda\delta_{a,c}\delta_{b,d} - \frac{\lambda}{\lambda + N + 1}\epsilon_a\epsilon_c\delta_{a,\bar{b}}\delta_{c,\bar{d}} \tag{2}$$

where $\bar{a} = 2N + 1 - a$, $\epsilon_a = 1$ for $1 \leqslant a \leqslant N$ and $\epsilon_a = -1$ for $N + 1 \leqslant a \leqslant 2N$.

With any integrable vertex model one can associate a local spin chain commuting with the corresponding transfer matrix whose matrix elements are given by ordered product of L factors $S_{ab}^{cd}(\lambda)$. As usual, the Hamiltonian is proportional to the logarithmic derivative of the transfer matrix at the regular point $\lambda = 0$, and in this case the expression is

$$H_{Sp(2N)} = \sum_{i=1}^{L} \left[\delta_{a,d}\delta_{c,b} - \frac{1}{N+1}\epsilon_a\epsilon_c\delta_{a,\bar{b}}\delta_{c,\bar{d}} \right] e_{ac}^{(i)} \otimes e_{bd}^{(i+1)} \tag{3}$$

where $\epsilon_{ab}^{(i)}$ is the elementary matrix $[e_{ab}]_{l,k} = \delta_{a,l}\delta_{b,k}$ acting on site i. We observe that the spectrum of $H_{SO}(J_1/3, J_1, J_1)$ matches that of $J_1 H_{Sp(4)} - J_1 L$. The $Sp(2N)$ Hamiltonian (3) is solvable by the Bethe ansatz [5, 6] and its eigenvalues $E(L)$ can be parametrized in terms of a set of variables $\lambda_j^{(a)}$, $j = 1, \ldots, m_a$ and $a = 1, \ldots, N$, satisfying the following Bethe equations:

$$\left[\frac{\lambda_j^{(a)} - \mathrm{i}\delta_{a,1}/\eta_a}{\lambda_j^{(a)} + \mathrm{i}\delta_{a,1}/\eta_a} \right]^L = \prod_{b=1}^{N} \prod_{k=1, k\neq j}^{m_b} \frac{\lambda_j^{(a)} - \lambda_k^{(b)} - \mathrm{i}C_{a,b}/\eta_a}{\lambda_j^{(a)} - \lambda_k^{(b)} + \mathrm{i}C_{a,b}/\eta_a}, \tag{4}$$

and the eigenvalues are given by

$$E(L) = -\sum_{i=1}^{m_1} \frac{1}{[\lambda_i^{(1)}]^2 + 1/4} + L \tag{5}$$

[1] In the notation of [13], equations (2.5)–(2.7), the $Sp(4)$ point corresponds to two strongly coupled XXZ chains with $J_1 = J_1^z = J_2/2 = J_2^z = 2/3$, $\lambda = 8/3$, $\Delta_1 = \Delta_2/2 = 1$.

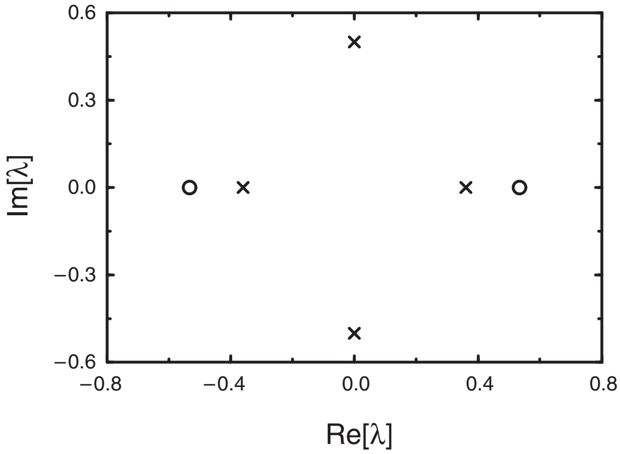

Figure 1. The ground state Bethe ansatz roots of the $Sp(4)$ spin chain for $L = 4$. The roots are: $\lambda_j^{(1)} = \{\pm 0.3602, \pm I/2\}$ (crosses); $\lambda_j^{(2)} = \{\pm 0.5337\}$ (circles).

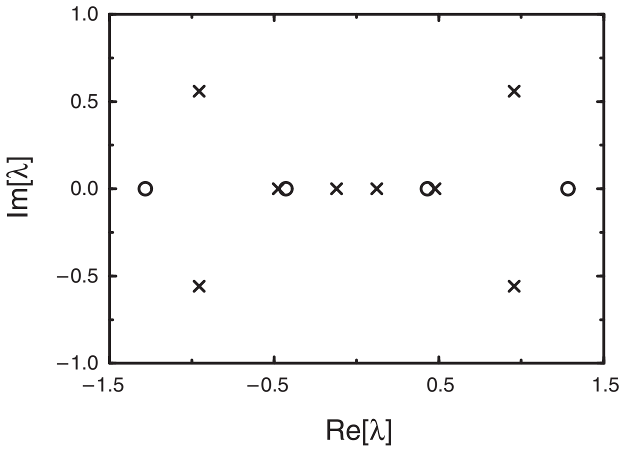

Figure 2. The ground state Bethe ansatz roots of the $Sp(4)$ spin chain for $L = 8$. The roots are: $\lambda_j^{(1)} = \{\pm 0.1226, \pm 0.4754, \pm 0.9568 + I * 0.5590\}$ (crosses); $\lambda_j^{(2)} = \{\pm 0.4295, \pm 1.2848\}$ (circles).

where C_{ab} is the Cartan matrix and η_a is the normalized length of the ath root of the $Sp(2N)$ algebra.

We start our study by considering first the $Sp(4)$ model, motivated by its direct relevance to the physics of spin–orbital systems. In fact, this is the simplest symplectic invariant system since $N = 1$ is equivalent to the isotropic six-vertex model. Later on we will show that the technicalities entering in the analysis of the $Sp(4)$ model can be easily generalized to include arbitrary $N > 2$. Essential to our study is determining the configurations of roots that describe the absolute ground state and the elementary excitations. This can be done by solving the Bethe equations (4), (5) for some values of L and comparing it with the exact diagonalization of the $Sp(4)$ spin chain. In figures 1 and 2 we exhibit the *ground state* Bethe ansatz roots for $L = 4$ and 8 together with the values of the roots, respectively. We see that the variables $\lambda_j^{(1)}$ (crosses) and $\lambda_j^{(2)}$ (circles) characterizing the *ground state* of the $Sp(4)$ model are described by strings with different lengths, namely

$$\lambda_j^{(1)} = \begin{cases} \xi_j^{(1)} \\ \xi_j^{(3)} \pm \mathrm{i}[\frac{1}{2} + \mathrm{O}(\mathrm{e}^{-\gamma L})] \end{cases} \qquad \lambda_j^{(2)} = \xi_j^{(2)} \tag{6}$$

with γ positive and where $\xi_j^{(\alpha)}$ ($\alpha = 1, 2, 3$) are real numbers. As usual, the low-lying excitations are described as the lack of certain $\lambda_j^{(1,2)}$ as compared with the ground state roots.

The important point here is to observe that the rapidities $\lambda_j^{(1)}$ are described in terms of two independent types of string, i.e. 1- and 2-strings. This feature should be contrasted with the behaviour of the Bethe ansatz roots of other fundamental integrable systems such as $SU(N)$ and $O(N)$ models [8, 16]. In fact, for the latter systems each ath root $\lambda_j^{(a)}$ characterizing the infinite-volume properties has a *unique* string length. To explore the consequences of the peculiar string configuration (6) we substitute it in the Bethe ansatz (4) and we obtain the following effective equations for the variables ξ_j^{α}:

$$L\big[\psi_{1/2}(\xi_j^{(\alpha)})\delta_{\alpha,1} + \psi_1(\xi_j^{(\alpha)})\delta_{\alpha,3}\big] = 2\pi Q_j^{(\alpha)} + \sum_{\beta=1}^{3}\sum_{k=1}^{N_\beta} \phi_{\alpha\beta}(\xi_j^{(\alpha)} - \xi_k^{(\beta)}) \tag{7}$$

where $\psi_a(x) = 2\arctan(x/a)$ and N_α is the number of roots $\xi_j^{(\alpha)}$. For L a multiple of four[2], the quantum numbers can be written as $Q_j^{(\alpha)} = -(N_\alpha - 1)/2 + j - 1$, with $j = 1, \ldots, N_\alpha$, and the matrix elements $\phi_{\alpha\beta}(x)$ are given by

$$\phi_{\alpha\beta}(x) = \begin{pmatrix} \psi_1(x) & -\psi_1(x) & \psi_{1/2}(x) + \psi_{3/2}(x) \\ -\psi_1(x) & \psi_2(x) & -\psi_{1/2}(x) - \psi_{3/2}(x) \\ \psi_{1/2}(x) + \psi_{3/2}(x) & -\psi_{1/2}(x) - \psi_{3/2}(x) & 2\psi_1(x) + \psi_2(x) \end{pmatrix}. \tag{8}$$

The ground state consists of a sea of 1- and 2-strings with $N_1 = N_2 = 2N_3 = L/2$. For large L, the roots $\xi_j^{(a)}$ are densely packed into its density distribution $\sigma(\xi_j^{(\alpha)}) = 1/L(\xi_{j+1}^{(\alpha)} - \xi_j^{(\alpha)})$ and the relations (7) in the $L \to \infty$ limit become integral equations for such densities. These integral equations are solved by elementary Fourier techniques and we find that

$$\sigma^{(1)}(x) = \frac{1}{2\cosh(\pi x)}, \qquad \sigma^{(2)}(x) = \frac{1}{6\cosh(\pi x/3)} \tag{9}$$

and

$$\sigma^{(3)}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sigma^{(1)}(y)\sigma^{(2)}(x-y)\,\mathrm{d}y = \frac{\frac{2}{\sqrt{3}}\sinh(2\pi x/3) - x}{6\sinh(\pi x)} \tag{10}$$

where we have emphasized the remarkable fact that $\sigma^{(3)}(x)$ is exactly the convolution of the densities $\sigma^{(1)}(x)$ and $\sigma^{(2)}(x)$. Recall that the function $\sigma^{(\alpha)}(x)$ is related to the continuous probability densities of finding the rapidity $\xi^{(\alpha)}$ with a given value x. We may therefore interpret $\sigma^{(3)}(x)$ as the probability for the sum of two independent events with probability $\sigma^{(1)}(y)$ at an arbitrary value y and $\sigma^{(2)}(x-y)$ at the complementary value $x - y$.

We now have the basic ingredients for investigating the thermodynamic limit properties. The ground state energy per site e_∞ is calculated by using equations (5), (9), (10) after replacing the sum by an integral. The final result is

$$e_\infty = -2\left[\frac{2\ln(2)}{3} + \frac{\pi}{9\sqrt{3}} - \frac{1}{3}\right]. \tag{11}$$

The low-lying excitations are obtained by inserting holes in the density distribution of $\xi_j^{(\alpha)}$, which means the removal of certain quantum numbers $Q_j^{(\alpha)}$ of the Bethe equations (7). The

[2] This emphasizes the $Sp(4) \supset Sp(2) \otimes Sp(2)$ symmetry.

necessary manipulations of these equations are standard [15, 16] and we find that the energy $\varepsilon^{(\alpha)}(\xi)$ and the momentum $p^{(\alpha)}(\xi)$ of one-hole excitation in the sea of $\xi_j^{(\alpha)}$, measured from the ground state, have the form

$$\varepsilon^{(\alpha)}(\xi) = \pi\sigma^{(\alpha)}(\xi), \qquad p^{(\alpha)}(\xi) = \int_{\xi}^{+\infty} \varepsilon^{(\alpha)}(x)\,\mathrm{d}x \tag{12}$$

where ξ is the α-hole rapidity. For the first two excitations one can easily eliminate the variable ξ leading us to the following dispersion relations:

$$\varepsilon^{(1)}(p) = \frac{\pi}{2}\sin(2p), \qquad \varepsilon^{(2)}(p) = \frac{\pi}{6}\sin(2p) \tag{13}$$

implying that these excitations are gapless and that their low-energy limits $\varepsilon^{(\alpha)}(p) \sim v^{(\alpha)}p$ are governed by distinct sound velocities, i.e. $v^{(1)} = \pi$ and $v^{(2)} = \pi/3$. The contribution to the total spin of each of these excitations is $\frac{1}{2}$, and therefore we shall interpret them as spinons propagating with different velocities which will separate in time.

Similar computation for the third excitation leads us to transcendental equations and an analytical expression for the dispersion relation is hard to obtain. However, it is possible to study the low-energy behaviour of such an excitation by expanding the density $\sigma^{(3)}(\lambda)$ in powers of $\mathrm{e}^{-\lambda}$. We see that the low-momenta regime is dominated by both sound velocities $v^{(1)}$ and $v^{(2)}$ and strictly in the $p \to 0$ limit the lowest one prevails. This massless excitation turns out to be a spinless mode whose speed of sound is $v^{(3)} = \pi/3$. At this point we note that recently the compound NaV_2O_5 has been modelled by an anisotropic/asymmetric spin–orbital model [18]. Remarkably, the three-particle continuum found above is in accordance with the excitation spectrum proposed in [18] to explain the optical properties of this material.

Next we would like to identify the underlying conformal field theory which describes the low-energy limit of the integrable $Sp(4)$ model. This can be investigated by analysing the behaviour of the finite-size spectrum [20] of the $Sp(4)$ spin chain. The type of critical behaviour expected here is that of a theory that is not strictly Lorentz invariant, because the sound velocities of the gapless excitations are not equal. This is the general behaviour of a fixed point of Tomonaga–Luttinger type and the predictions of the conformal invariance for the finite-size behaviour [20] are still applicable after slight modifications [21]. For example, one expects the ground state energy $E_0(L)$ of the $Sp(4)$ spin chain to behave for large L as

$$\frac{E_0(L)}{L} = e_\infty - \frac{\pi}{6L^2}\sum_{a=1}^{3} v^{(a)}c^{(a)} \tag{14}$$

where $c^{(a)}$ and $v^{(a)}$ are the central charge and the sound velocity associated with each possible massless degree of freedom.

In order to get a rough idea of the behaviour of the ground state finite-size correction, one can apply the root density method [22] to the string Bethe ansatz equations (7). Within this approach we find that each ath massless contributes with the value $c^{(a)} = 1$ for the finite-size correction (14). However, it is well known that the string assumptions may not give the correct finite-size behaviour because the complex part of the roots can also contribute to the term $1/L^2$ as well; see for instance [23]. This means that the true contribution of the third mode to the finite-size correction (14) could be different from the value $c^{(3)} = 1$. To investigate this possibility we have solved numerically the original Bethe ansatz equations (4) up to $L = 36$ which allows us to compute the sequence

$$C_{ef}(L) = -\left[\frac{E_0(L)}{L} - e_\infty\right]\frac{6L}{\pi v^{(2)}} \tag{15}$$

Table 1. The finite-size sequence $C_{ef}(L)$ and its extrapolation ('Extr.') c_{ef}.

L	$C_{ef}(L)$
12	4.074 654
16	4.046 720
20	4.033 092
24	4.025 305
28	4.020 376
32	4.017 024
36	4.014 620
Extr.	4.001 (±2)

whose $L \to \infty$ limit[3] is $c_{ef} = 3c^{(1)} + c^{(2)} + c^{(3)}$.

In table 1 we exhibit the sequence (15) for several values of L together with its extrapolated value. We see that c_{ef} is likely to be $c_{ef} = 4$ instead of $c_{ef} = 5$ as predicted by the string hypothesis; that is, $c^{(a)} = 1$ for each $a = 1, 2, 3$. This means that the imaginary parts of the roots $\lambda_j^{(1)}$ have conspired together to cancel the $1/L^2$ correction proportional to the third mode. It is therefore tempting to think of the third excitation as a composite state of two elementary spinons which does not contribute to the low-energy limit. As a consequence of that, the continuum limit is described by a field theory with central charge $c = 1 \otimes 1$ which is different from the conformal anomaly of the $Sp(4)$ WZW model.

Similar analysis can be performed for the excited states $E_i(L)$, which enables us to determine the corresponding conformal dimension by extrapolating the finite-size gap [21]:

$$\frac{E_i(L)}{L} - \frac{E_0(L)}{L} = \frac{2\pi}{L^2} \sum_{a=1}^{3} v^{(a)} X_i^{(a)}. \tag{16}$$

In table 2 we show the corresponding sequence $X_{ef}(L) = [E_1(L) - E_0(L)]\frac{L}{2\pi v^{(2)}}$ for the lowest excited state of the $Sp(4)$ spin chain. As before, it is useful to compute the same amplitude by the root density method [22]. In this case we found that as $L \to \infty$, $X_{ef}(L)$ produces the value $x_{ef} = \frac{3}{4} + \frac{1}{4} = 1$. We see that the extrapolated value exhibited in table 2 agrees well with the analytical estimate of x_{ef}, leading us to interpret the operator content of the lowest excitation as $X_1^{(1)} = X_1^{(2)} = 1/4$. This information together with the central charge behaviour suggests that each elementary spinon mode $a = 1, 2$ is well described in terms of the product of two Majorana fields. Therefore, the underlying field theory of the $Sp(4)$ spin chain is likely to be represented in terms of four Majorana fermions, perturbed by non-Lorentz interaction terms that just renormalize the sound velocities, rather than be given by a $Sp(4)$ WZW theory.

Let us now turn to the problem of extending our results for general $N > 2$. The Bethe ansatz equations are parametrized by N different types of roots, and it turns out that the first $N - 1$ roots are given by both 1- and 2-strings while the last one behaves like 1-strings. If we characterize the centre of the strings by $\xi_j^{(\alpha)}$, $\alpha = 1, \ldots, 2N-1$, we find that the corresponding *ground state* density distributions are

$$\sigma^{(\alpha)}(x) = \begin{cases} \dfrac{1}{N} \dfrac{\sin(\pi\alpha/N)}{\cosh(2\pi x/N) - \cos(\pi\alpha/N)}, & \alpha = 1, \ldots, N-1 \\ \dfrac{1}{2(N+1)} \dfrac{1}{\cosh[\pi x/(N+1)]}, & \alpha = N \end{cases} \tag{17}$$

[3] Recall that we have two possible momenta scales, π or $\pi/3$, and one needs to choose a way to normalize the sequence (15). Here we choose to do it via $v^{(2)} = v^{(3)} = \pi/3$.

Table 2. The finite-size sequence $X_{ef}(L)$ and its extrapolation ('Extr.') x_{ef}.

L	$X_{ef}(L)$
12	1.000 985
16	1.004 993
20	1.006 011
24	1.006 017
28	1.005 692
32	1.005 272
36	1.004 845
Extr.	1.006 (± 2)

while for $\alpha = N+1, \ldots, 2N-1$ they are given by the convolution $\sigma^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sigma^{(2N-\alpha)}(y)\sigma^{(N)}(x-y)\,\mathrm{d}y$.

The excitations are gapless, consisting of $N-1$ generalized $SU(N)$ spinons [16] propagating with velocity $v^{(\alpha)} = \frac{2\pi}{N}$ for $\alpha = 1, \ldots, N-1$ and one standard spinon whose speed of sound is $v^{(N)} = \frac{\pi}{N+1}$. In addition, we have $N-1$ composite modes made by the convolution between the first $N-1$ spinons with the Nth excitation. For $N > 2$, numerical results for large L become difficult to obtain since the number of roots to be determined grows rapidly with both N and L. However, for $N = 3$ and small $L \sim 18$, our numerical analysis is consistent with the fact the only modes contributing to the low-energy properties are the spinons, each one with $c = 1$. All of these results seem to be strong evidence that the continuum limit of such $Sp(2N)$ integrable models can indeed be described in terms of $2N$ Majorana fermions.

In conclusion, we have studied the excitation spectrum of the simplest integrable $Sp(2N)$ spin chain. Contrary to common belief, this system is not the lattice realization of the $Sp(2N)$ WZW conformal theory. Our study indicates that the nature of the excitations in spin–orbital systems can be rather involved. In fact, the isotropic point $J_0 = J_1 = J_2$ is known to have three basic excitations [16], being the lattice realization of a $SU(4)$ WZW field theory [7]. However, the anisotropic point $J_0/J_1 = J_0/J_2 = 1/3$ has only two independent excitations and one composite mode that do not contribute to the low-energy limit, and it is described by a $c = 2$ conformal field theory. This work prompts us to ask some questions that may open up new interesting avenues. What is the nature of the excitations of the spin–orbital model (1) in the crossover regime $1/3 \leqslant J_0/J_1 = J_0/J_2 \leqslant 1$? What is the mechanism that made one of the excitations become a composite state? Which is the integrable lattice $Sp(2N)$ model whose continuum limit corresponds to the $Sp(2N)$ WZW theory?

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